RA 18
(there was no RA 17)

10.1 Sum and Product Rules

- The product rule provides a way to count sequences.
- The Product Rule:
  - Let $A_1, A_2, \ldots, A_n$ be finite sets, then:

$$|A_1 \times A_2 \times \ldots \times A_n| = |A_1| \cdot |A_2| \cdot \ldots \cdot |A_n|$$

$\boxed{\text{Ex}}$ How many string of length 4 are there over the alphabet $\{a, b, c\}$?

$3 \cdot 3 \cdot 3 \cdot 3 = 81$

How many strings of length 4 are there over the alphabet $\{a,b,c\}$ that end with character c.

$3 \cdot 3 \cdot 3 \cdot 1 = 27$

- The sum rule:
  - Consider n sets, $A_1, A_2, \ldots, A_n$. If the sets are pairwise disjoint (which means that $A_i \cap A_j = \emptyset$ for $i \neq j$)

    then:

$$|A_1 \cup A_2 \cup \ldots \cup A_n| = |A_1| + |A_2| + \ldots + |A_n|$$

$\boxed{\text{Ex}}$ How many 6-bit strings begin and end with a 1, or start with 00.

↳ Begin and End with 1: $1 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 1 = 16$

↳ Start with 00: $1 \cdot 1 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 16$

↳ Add the two: $16 + 16 = \boxed{32}$

10.3 The generalized product rule

- The generalized product rule says that in selecting an item from a set, if the number of choices at each step does not depend on previous choices made, then the number of items in the set is the product of the number of choices in each step.

- Generalized product rule:
  - Consider a set S of sequences of k items. Suppose:
    - n_1 choices are available for the first item
    - for every possible choice for the first item, n_2 choices are available for the second item.
    - for every possible choice for the 1^{st} and 2^{nd} item, n_3 choices are available for the third item.

    ⋮

    - For every possible choice for the $k-1$ items, n_k choices are available for the k^{th} item.
  - Then $|S| = n_1 \cdot n_2 \cdot \ldots \cdot n_k$

10.4 Counting Permutations

- An r-permutation is a sequence of r items with no repetitions, all taken from the same set.

  ↳ order does matter

- The number of r-permutations from a set with n-elements.

  ↳ Let r and n be positive integers with $r \leq n$. The number of r-permutations from a set is denoted by $P(n,r)$. $P(n,r) = \frac{n!}{(n-r)!}$

- A permutation (without the parameter r) is a sequence that contains each element of a finite set exactly once. For example, the set $\{a,b,c\}$ has six permutations.

1. (a,b,c)	4. (a,c,b)
2. (b,a,c)	5. (b,c,a)
3. (c,a,b)	6. (c,b,a)

- The number of permutations of a finite set is $P(n,n) = n!$

Ex A wedding party consisting of a bride, a groom, 2 bridesmaids, 2 groomsmen line up for a photo. How many ways can they line up so that bride is next to groom?

2 ways for bride and groom: [Bride][Groom] or [Groom][Bride]

↳ they must be together so treat them as one entity now.

↳ Now we have 5 entities and on entity can be in 2 "states", so:

$[5 \cdot 4 \cdot 3 \cdot 2 \cdot 1] \times 2 = \boxed{240}$

10.5 Counting Subsets

- A size r subset is called an r-subset.
- An r-subset is sometimes referred to as an r-combination.

Ex Let $S = \{a, b, c\}$

1. Is (b, a) a 2-permutation or a 2-subset from S?

   ↳ 2-permutation.

2. Is $\{b, a\}$ a 2-permutation or a 2-subset from S?

   ↳ 2-subset

3. How many different 2-permutations from S are there?

   ↳ 6

4. How many different 2-subsets are there?

   ↳ 3

- Counting subsets 'n choose r' notation.
  - The number of ways of selecting an r-subset from a set of size n is:

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

- Also note:

$$\binom{n}{n-r} = \frac{n!}{(n-r)!\left[(n)-(n-r)\right]!} = \frac{n!}{(n-r)!\,r!} = \binom{n}{r}$$

- An equation is called an <u>identity</u> if the equation holds for all values for which the expressions in the equation are well defined.